Extended Finite Element Method Lab

Mohammad Shahid

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Abstract

A software named as crackcomput, based on XFEM and Xcrack libraries was used to carry out various experiments. In the first experiment convergence properties of the method was analyzed on an elasto-statics problem. Then in the second experiment, stress intensity factors were compared to simplified analytical results. And in the final experiment a Brazilian fracture test was conducted. All the experiments were done on a 2D domain under plane-stress assumption. Moreover, the material was considered to be elastic and isotropic.

1 Introduction

There are a large number of physical problems where the field quantities charge drastically over quite smaller length scales as compared to the domain size. The resulting solutions involves local discontinuities, high gradients as for example in case of shocks, boundary layers, etc., in fluid mechanics, and, cracks, dislocations, voids, etc., in solid mechanics.

In order to approximate such problems the classical Finite Element Method (FEM) relies on the construction of meshes that align with the discontinuities and are refined near high gradients. And in case of a temporaly evolving discontinuities, the classical FEM requires re-meshing and projection of the field to the newly generated mesh, which directly adds to the computational cost. To abjure these problems, XFEM provides an alternate approach to model such problems by enriching the approximation spaces of the FEM such that the non-smooth properties are accounted for correctly, independent of the mesh. As a consequence, simple, fixed meshes can be used throughout the simulation, and mesh construction and maintenance are reduced to a minimum.

2 Convergence Analysis

In this exercise, the mode I crack opening for an infinite plane is studied. The goal of this exercise was to measure the error between the exact solution and the numerical solution as well as the convergence rate for different simulation parameters. To emulate the infinite problem, a square shaped domain is used as shown in figure 1.

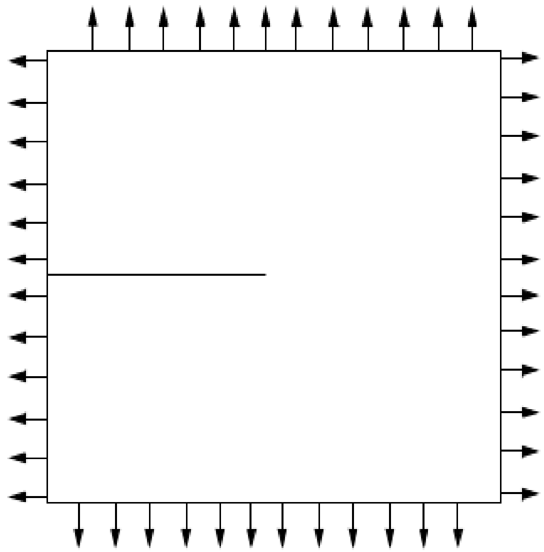

Figure 1: Crack in an infinite plane, modeled using stress of the exact solution at the boundary

Simulations were performed using different number of elements for both type of enrichment (scalar and vector) for different enrichment radii, the details of which is tabulated in Table 1

Enrichment Type & Radius	Vector						Scalar		
	Element Order = 1			Element Order = 2			Element Order = 1		
	No. of Elements			No. of Elements			No. of Elements		
	100	361	1369	100	361	1369	100	361	1369
0.2	0.2559	0.1391	0.0799	0.0429	0.0189	0.0059	0.2370	0.1208	0.0646
0.4	0.2071	0.1053	0.0569	0.0240	0.0082	0.0023	0.1807	0.0819	0.0424
0.6	0.1748	0.0815	0.0451	0.0156	0.0043	0.0012	0.1434	0.0576	0.0315
0.8	0.1455	0.0699	0.0372	0.0010	0.0030	0.0008	0.1102	0.0454	0.0240

Table 1: Brief Overview of the various cases with corresponding results

In order to elucidate the effect of various parameters considered in the study on the error, we plot variation of error with mesh size for all the cases listed in the Table 1.

2.1 Comparison of errors for different radii of enrichment for Vector Enrichment

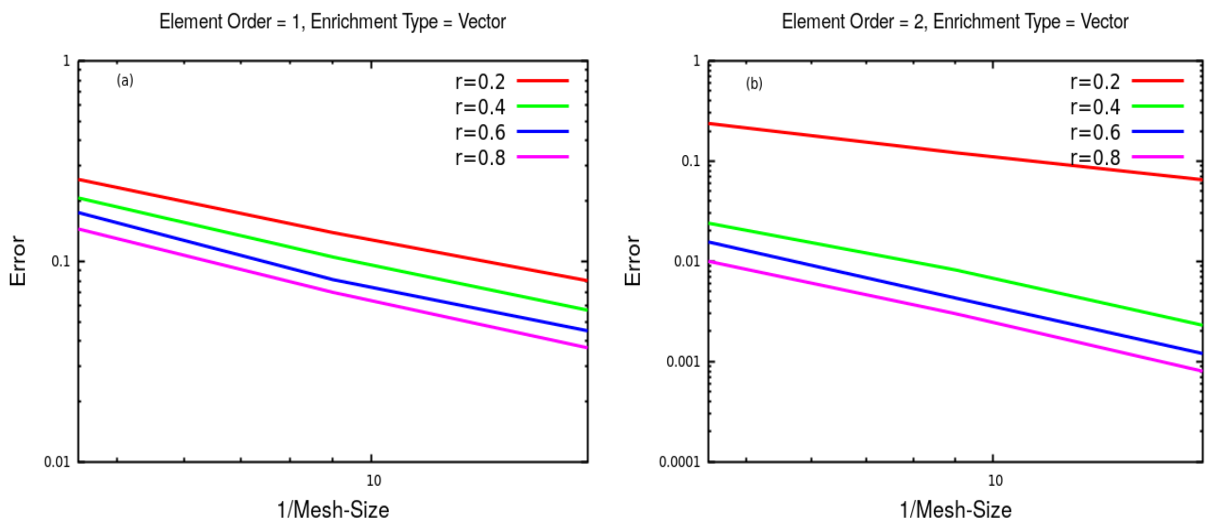

Figure 2: Error vs 1/mesh-size for Vector enrichment for different radii of enrichment using (a) First order elements and (b) Second order elements

It is evident from figures 2(a) and 2(b) that as we increase the radius of enrichment the error between the approximated solution and the exact solution decreases as more nodes are getting enriched near the vicinity of the crack tip thereby increasing the accuracy of the solution. The effect of increasing the radius of enrichment is more pronounced in case of second order elements.

The effect of using higher order elements is more clearly described in figure 3, where we can clearly observe that increasing the order of elements significantly improves the accuracy of the solution for a particular radius of enrichment. In general, error can be written in terms of mesh-size h as:

$$\epsilon_m \propto h^{(p+1-m)} \tag{1}$$

where, p and m is the order of the element and the error considered ($m = 0$ for displacement error and $m = 1$ for strain error). Hence, with higher order elements XFEM gives much faster convergence and better accuracy of the solution. Had it been the case of classical FEM, we wouldn't have gain any improvement in accuracy of the solution near the crack tip, as the error in case of classical FEM depends on mesh-size as h:

$$\epsilon_m \propto h^{min(r-m,p+1-m)} \tag{2}$$

So, if we are in a corner the classical FEM does the worst job in approximating the solution. The most remarkable feature of the XFEM is that it kills the singularity produced by the term $(r - m)$, thereby giving much accurate solution with increasing order of the element.

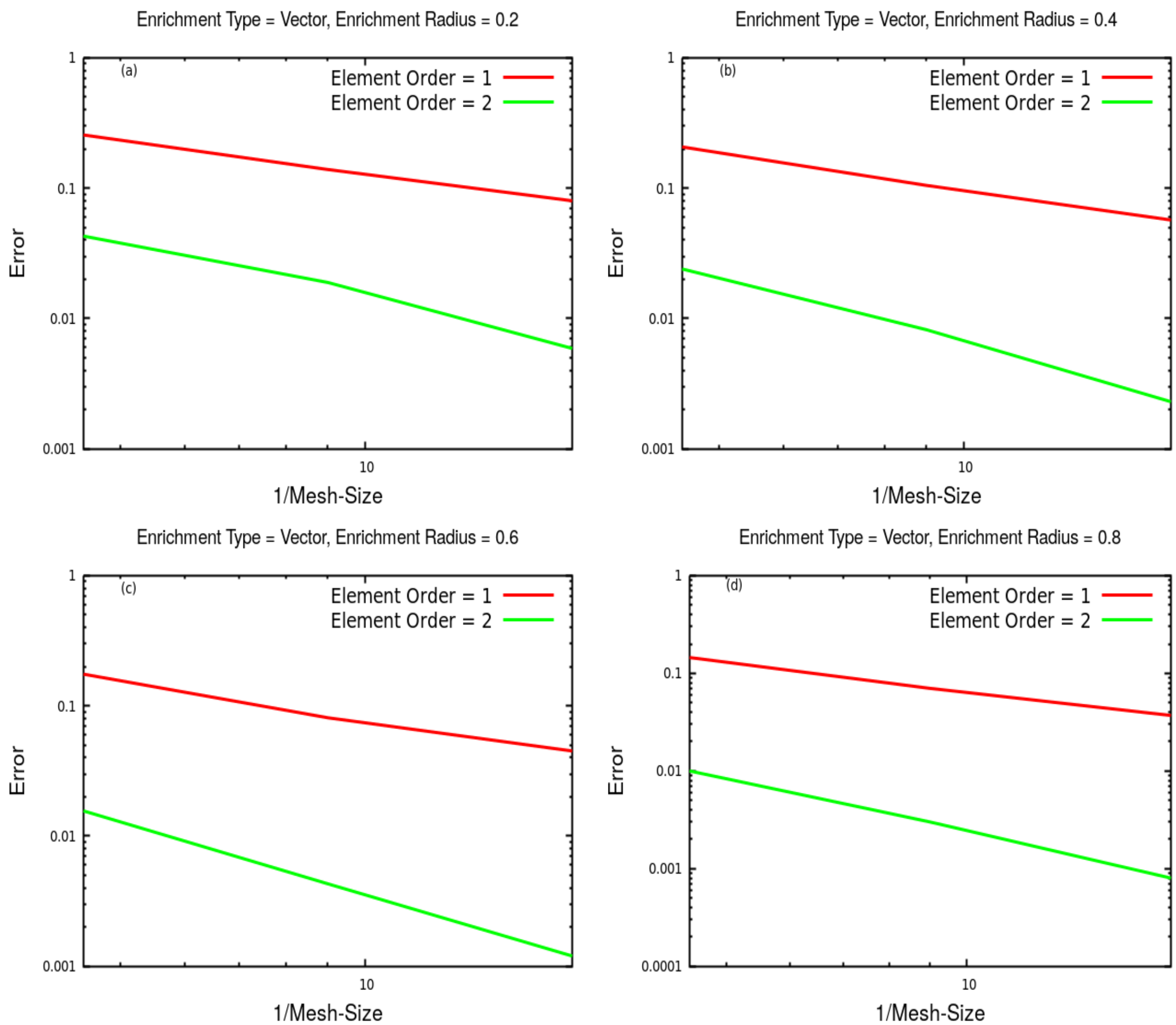

Figure 3: Error vs 1/mesh-size for Vector enrichment for different radii of enrichment using First and Second order elements

2.2 Comparison of errors between Vector and Scalar enrichment for different radii of enrichment using first order elements

From figure 4 it is obvious that the scalar enrichment gives a slightly better approximation of the solution as compared to vector enrichment. It is due to the fact that, in case of 2D, the scalar enrichment requires 8 additional degrees of freedom, as compared to 2 additional degrees of freedom in case of vector enrichment, to model the discontinuity. However, the accuracy comes with the cost of more computational time, and one can clearly observe from figure 4 that the disparity in errors for both type of enrichment is not substantial. So it would be a better compromise to use vector enrichment instead of scalar enrichment for modeling crack problems, as it is efficacious in terms of computational cost without substantially compromising the accuracy of the solution.

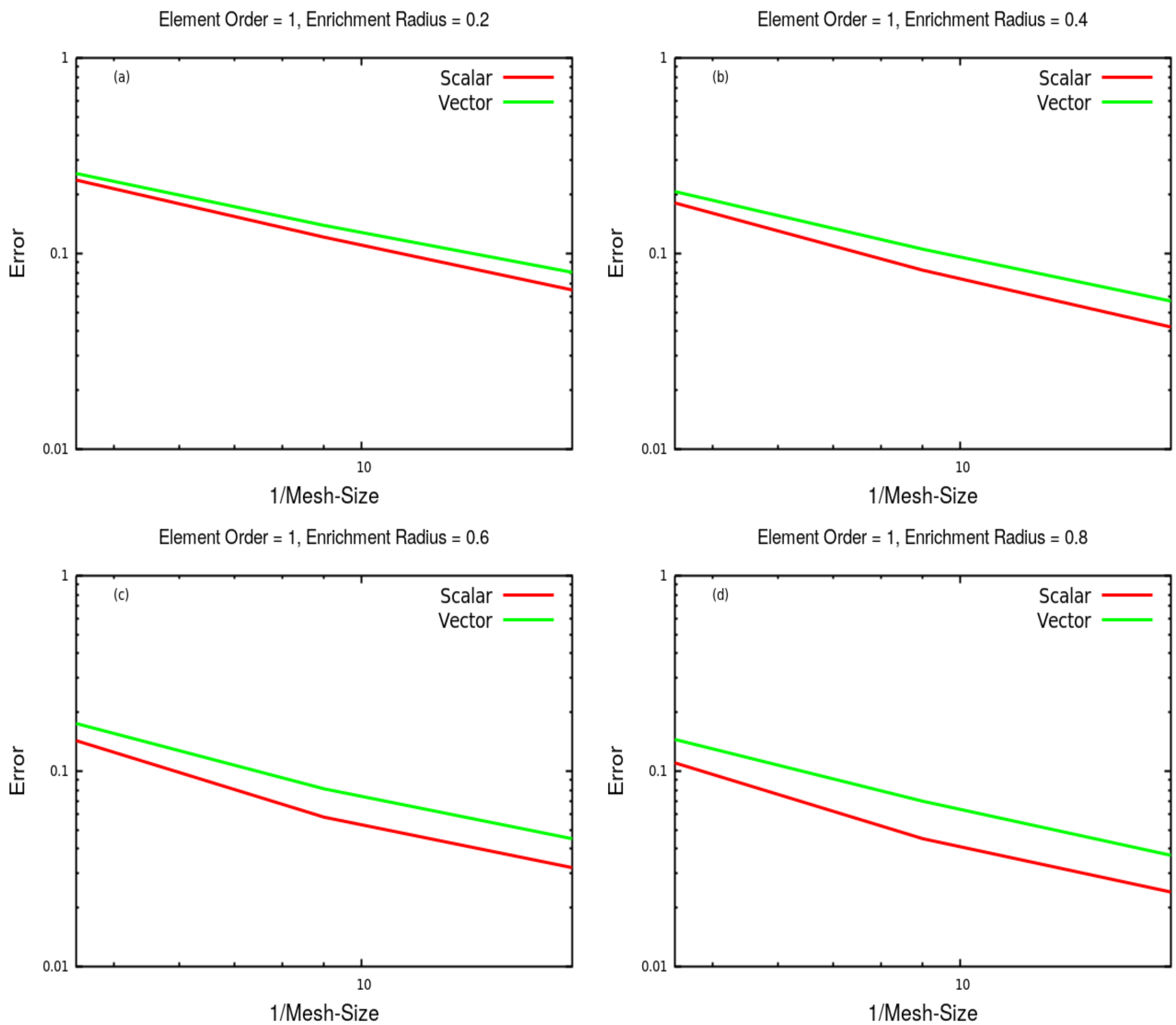

Figure 4: Error vs 1/mesh-size for Vector and Scalar enrichment for different radii of enrichment using First order elements

3 Crack in a beam

In this exercise, stress intensity factor at the crack tip is computed and compared to the analytical results on an idealized situation. The crack in the beam is modeled in two dimensions as in figure 2. The expressions for the energy release rate during crack growth can be found in many fracture mechanics textbooks. The goal of this exercise is to compare the analytical results with computed stress intensity factor at the crack tip.

The stress intensity factor extraction capacity of crackcomput is used to compute the stress intensity factor. The parameters such as mesh size, enrichment, the size of the integral zones, etc. were chosen to give enough precision as crackcomput implement a domain integral method for the extraction of stress intensity factors.

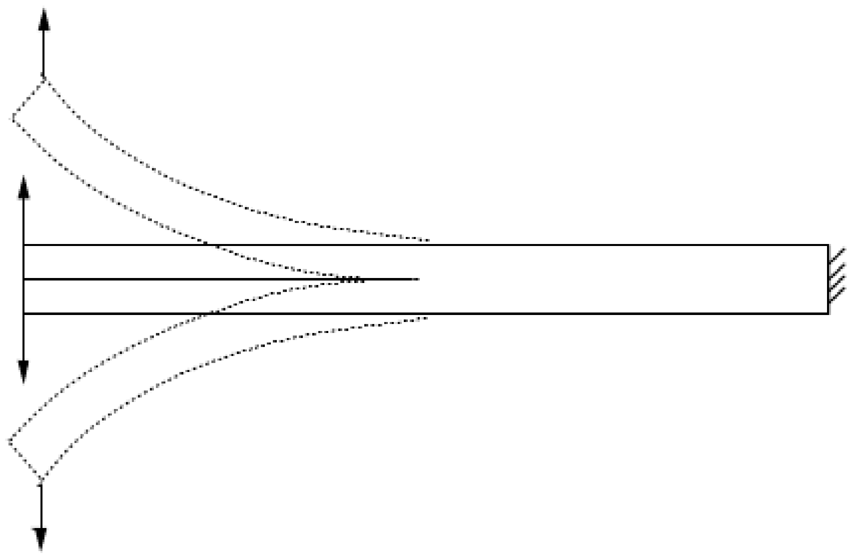

Figure 5: Crack in a beam

Consider a body with a boundary contour Γ subjected to an energy input increment (dU) due to an external loading. Assuming the body to be elastic, its mechanical behaviour can be characterized by a linear load-displacement relationship, $P = f(\mu)$. The energy change in a loaded plate occurs due to the displacements arising from the fracture area change $dA = d\partial a$ for a constant thickness B and variable crack length a. Thus, the input energy change (dU_1) is divided into the change in dissipated energy (dU_2) as heat which arises due to the irreversible process during plastic flow, the change in stored energy (dU_3), and the change in kinetic energy (dU_4) of the system. Consequently, the conservation of energy change due to the displacements arising from the fracture area change dA can be defined as

$$\frac{dU_1}{dA} - \frac{dU_2}{dA} = \frac{dU_3}{dA} + \frac{dU_4}{dA} \tag{3}$$

For a growing crack, $\frac{dU_2}{dA}$ is the energy dissipated in propagating fracture over an increment of area dA which is referred to as the fracture resistance R. On the other hand, $\frac{dU_1}{dA} - \frac{dU_2}{dA}$ can be defined as the net energy input. The energy release rate G_i, where i refers to the loading modes or their combinations, can be defined as

$$G_i = \frac{dU_1}{dA} - \frac{dU_3}{dA} \tag{4}$$

For a stationary body, the energy release rate criterion for crack tip instability is

$$G_{iC} \geq R \tag{5}$$

Assuming if the cracked plate is subjected to an external load P and the growth of crack is slow, then the loading points undergo a relative displacement $d\mu$ perpendicular to the crack plane and the crack length extends to an amount da. Consequently, the work done for such an increment in displacement can be given by

$$\frac{dU_1}{da} = P\frac{\mu}{da} \tag{6}$$

Considering only the mode I loading and the linear behaviour. the energy stored due to tension can be obtained using

$$U_3 = \frac{1}{2}P\mu \tag{7}$$

$$\implies \frac{dU_3}{da} = \frac{P}{2}\frac{d\mu}{da} + \frac{\mu}{2}\frac{dP}{da} \tag{8}$$

$$\implies G_I = \frac{1}{2B}\left(P\frac{d\mu}{da} - \mu\frac{dP}{da}\right) \tag{9}$$

For essentially elastic response, the displacement takes the form

$$\mu = PC \tag{10}$$

For the present problem for constant displacement: $P = \frac{\mu}{C} = -\frac{9\mu EI}{2a^4}$ for constant μ. Hence, $G_I = -\frac{\mu}{2B}\frac{dP}{da} = \frac{9\mu^2 EI}{4Ba^4}$, and since $I = \frac{1}{12}Bh^3$, therefore $G_I = \frac{3\mu^2 Eh^3}{16a^4}$.
As for Mode I the loading is given as $K_I = \sqrt{EG}$.

$$\implies K_I = \frac{\sqrt{3}\mu Eh^{3/2}}{4a^2} \tag{11}$$

Substituting the values of μ, E, h and a, we get

$$\implies K_I = 1.36 \times 10^{-5}\frac{N}{mm^{3/2}} \tag{12}$$

Several runs were taken inorder to obtain the suitable mesh-size for rest of the cases. The mesh-size is chosen such that the stress intensity factor for Mode II (K_{II}) converges to zero. The parameters chosen for the numerical study are listed in the table 2.

Enrichment Type	Vector
Element Order	1
Enrichment Radius	0.5
Points on the lip	15
Youngs Modulus	1
Poisson ratio	0

Table 2: Parameters chosen for the numerical study

L/h	$K_I (\times 10^{-4})$
10	53.70
20	14.30
30	06.51
40	03.70
50	02.39
60	01.67
70	01.23
80	00.94
90	00.75
100	00.61
110	00.50
120	00.42
130	00.36
140	00.31
150	00.27

Table 3: Computed values of K_I for different L/h ratios

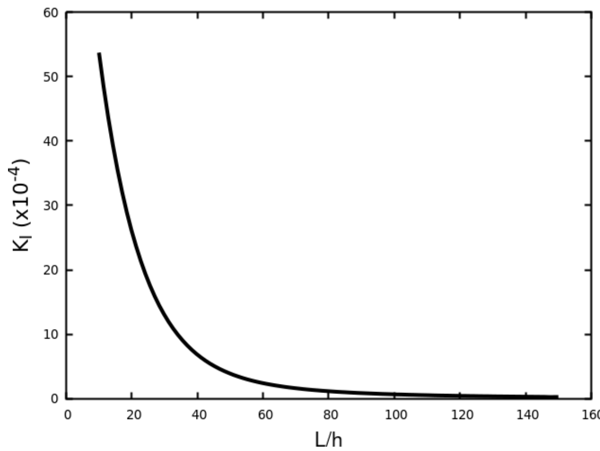

Figure : Variation of K_I vs L/h

In all of the cases, the length of the crack was kept as half of the length of the beam and the length of the beam is varied keeping the thickness fixed. The number of elements were chosen in a way to get the similar mesh-size in all the cases.

4 Brazilian Test

Concrete is the most widely used construction material. It is a composite material that is complex over a wide range of length scales, ranging from nanometers to meters. In general, failure in concrete structures, highway pavement, and dams, among other construction projects, can be attributed to some form of tensile stress. The tensile strength (TS) is determined by the direct tensile test. The Brazilian test is an experimental test that permits an indirect inference of the TS even though it measures the indirect tensile strength (ITS). The ITS is related to the compression strength (CS), water-cement ratio, and age of the concrete, among other factors (Zain et al. 2002).

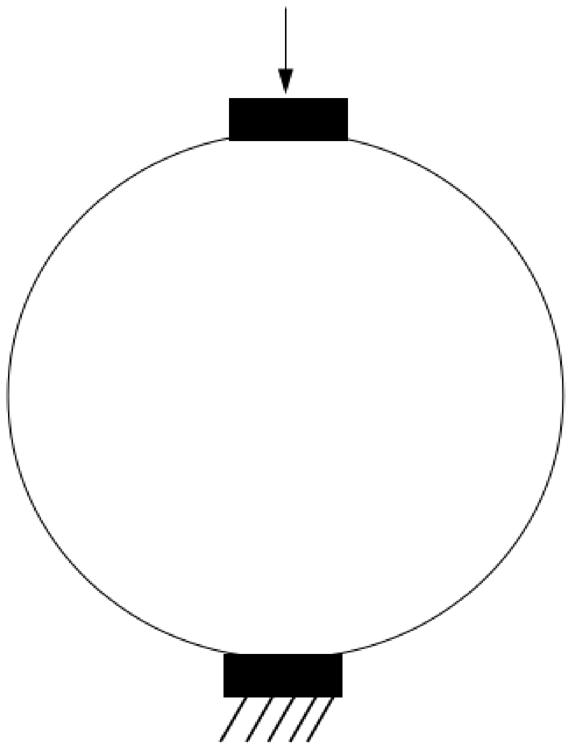

Figure 6: Crack in a beam

In the Brazilian test a circular disk is compressed in a diametrical plane between surfaces that apply concentrated loads. This test was developed to overcome the difficulties associated with preforming a direct uni-axial tensile test. It was used by Hondros to determine the elastic properties of concrete. If a circular disk of radius R and unit thickness is compressed across a diameter by line loads W then the stresses on the diameter is given by Jaeger and Cook as

$$\sigma_y = \frac{W}{\pi R} \tag{13}$$

$$\sigma_x = \frac{-W(3R^2 + x^2)}{\pi R(R^2 - x^2)} \tag{14}$$

Crack growth in real structures is often stable and develops slowly with initially partial and then total stress-relaxation at the crack tip. Only in some particular cases, do crack formation and/or propagation occur suddenly with a catastrophic drop in the load carrying capacity. When the fracture phenomenon in extremely brittle and only energy dissipation on the crack surfaces occurs, LEFM can properly describe the catastrophic structure behaviour. For extremely brittle cases a snap-back instability in the equilibrium path occurs and the load-deflection softening branch assumes a positive slope. This means that both load and deflection must decrease to obtain slow and controlled crack propagation, whereas in normal softening only the load must decrease. The snap-back instability was originally detected by Fairhurst et al. (1971) in the compressive behaviour of the rock.

The goal of this exercise is to reproduce the experiment, where a crack appear in the center of the sample in the vertical direction and then propagate vertically until total failure of the sample and plot the displacement/force curve at the loading point while the crack propagate under the assumption that the crack advance when the stress intensity factor KI reach a critical value K_{Ic}. Several runs were taken with different crack lengths to calculate the loading required for crack to propagate.Table 4 summarizes the list of simulation parameters and the computed results.

Crack Length	**Displacement for F=1N (x10^{-9})**	$\mathbf{K_I}$	**Critical Force**	**Actual Displacement (x10^{-8})**
0.1	5.31	0.069	14.51	7.70
0.2	5.35	0.101	9.89	5.29
0.3	5.42	0.132	7.60	4.12
0.4	5.52	0.164	6.11	3.37
0.5	5.69	0.199	5.03	2.86
0.6	5.94	0.236	4.23	2.51
0.7	6.31	0.273	3.67	2.31
0.8	6.83	0.289	3.46	2.36
0.9	7.51	0.233	4.30	3.23
0.95	7.87	0.143	6.97	5.49

Table 4: List of various cases along with computed forces and displacements

The Stress Intensity Factor K_I is related to force as $K_I = \alpha F$. As the analysis is performed for a load of $1N$, hence $K_I = \alpha$. To obtain critical force the value of critical stress intensity factor (K_{I_c}) is taken as 1. The obtained value of K_I is inverted to get the actual force which when multiplied with the obtained displacement gives the true displacement. The obtained actual force and displacement is plotted in figure 7.

Without crack the behaviour is expected to be linear and the specimen will have high load bearing capacity. With the initiation of the crack the specimen will start to loose its strength and will undergo snap-back phenomenon as demonstrated by the force-displacement curve. The loss of energy of the system due to change in loading can be measured by calculating the difference in the areas under the force-displacement curve between the two stages.

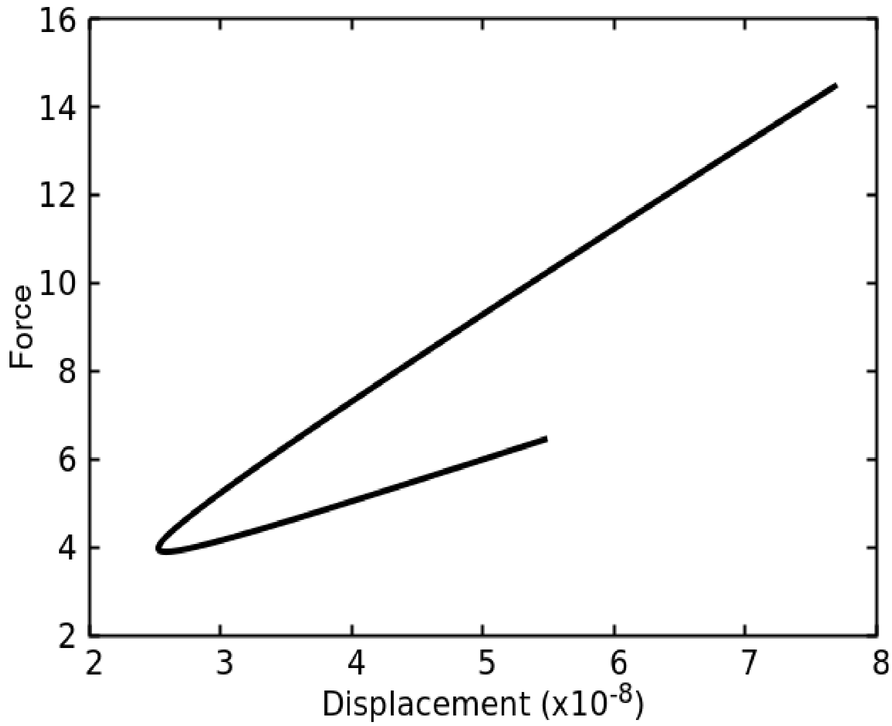

Figure 7: Force vs Displacement curve

5 Conclusions

Convergence analysis experiment shows that it is efficacious to use vector enrichment to approximate the solution of the problem without substantially compromising on the accuracy because of the lower computational time as compared to the scalar enrichment for a given order of element.

The study of crack in a beam gives an idea about the Stress Intensity Factor K_I when the 2-D model is manipulated to produce 1-D behavior. Hence the study can be extended for the evaluation of stress intensity factor for 2-D model, for the mixed mode and other discontinuities.

The study of the Brazilian Test gives an idea of the Snap-Back behavior of the sample under the specified loading condition.

References

[1] A. Carpinteri. "Interpretation of the Griffith instability as a bifurcation of the global equilibrium. Application of fracture mechanics to cementitious composites." (Edited by S.P. Shah), Martinus Nijhoff Publishers, Dordecht, 287-316.

[2] A. Carpinteri and G. Colombo. "Numerical analysis of catastrophic softening behaviour (snap-back instability)." *Computers and Structures* 31, 607-636 (1989)

[3] A. Carpinteri and I. Monetto. "Snap-back analysis of fracture evolution in multi-cracked solids using boundary element method." *Int. J. Frac.* 98, 225-241 (1999)

[4] C. Fairhurst, J. A. Hudson and E. T. Brown, "Optimizing the control of rock failure in servo-controlled laboratory tests." *Rock Mech.* 3, 217-224 (1971)

[5] D.F. Malan, J.A.L. Napier and B.P. Watson. "Propagation of fractures from an interface in a Brazilian test specimen." *Int. J. Rock Mech. Min. Sci. & Geomech. Abstr.* 31-6, 581-596 (1994)

[6] J. Claesson and B. Bohloli. " Brazilian test: stress field and tensile strength of anisotropic rocks using an analytical solution." *Int. J. Rock Mech. & Min. Sci.* 39, 991-1004 (2002)

[7] N. Perez. "Fracture Mechanics". Kluwer Academic Publishers.

[8] S. Pommier, A. Gravoiul, A. Combescure and N. Moës. "Extended Finite Element Method for Crack Propagation", John-Wiley & Sons (2011)

[9] S. Mohammadi. "Extended Finite Element Method for Fracture Analysis of Structures" Blackwell Publishing (2008)

[10] T.P. Fries, T. Belytschko. "The extended/generalized finite element method: An overview of the method and its applications." *Int. J. Num. Meth. Engg.* 1-6 (2000)

[11] T. Tanabe, A. Itoh and N. Naoshi. "Snapback failure analysis for large scale concrete structures and its application for shear capacity study of coloumns." *J. Adv. Concrete Tech.* 2(3), 275-228 (2004)